Due Sat

2.5 – Solutions by Substitution

We'll use substitutions to solve DEs (and IVPs) of the following types.

6. $(y^2 + yx)\,dx + x^2\,dy = 0$

Bernoulli equation

16. $\dfrac{dy}{dx} - y = e^x y^2$ $\leftarrow y^n$

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

Sub: $u = y^{1-n}$

Here $u = y^{-1}$

$y = \dfrac{1}{u}$

$\dfrac{dy}{dx} = -\dfrac{1}{u^2}\dfrac{du}{dx}$

26. $\dfrac{dy}{dx} = \sin(x + y)$

Sub: $u = x + y$

Form: $u = ax + by + c$

$\dfrac{du}{dx} = 1 + \dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} - 1$

$$\frac{du}{dx} - 1 = \sin u$$

$$\frac{(1-\sin u)}{(1-\sin u)}\frac{du}{1+\sin u} = dx$$

$$\frac{1-\sin u}{\cos^2 u}\,du = dx$$

$$(\sec^2 u - \tan u \sec u)\,du = dx$$

$$\tan u - \sec u = x + C$$

$$\boxed{\tan(x+y) - \sec(x+y) = x + C}$$

$u = y^{1-n}$

$n = 2$

16. $\dfrac{dy}{dx} - y = e^x y^2$

$u = y^{1-2} = \dfrac{1}{y}$

$y = \dfrac{1}{u}$

$\dfrac{dy}{dx} = -\dfrac{1}{u^2}\dfrac{du}{dx}$

$$-\frac{1}{u^2}\frac{du}{dx} - \frac{1}{u} = e^x \frac{1}{u^2}$$

$$\frac{du}{dx} + u = -e^x$$

Linear, $\mu = e^x$

$$\frac{d}{dx}\left(e^x u\right) = -e^{2x}$$

$$e^x u = -\tfrac{1}{2}e^{2x} + C_1$$

$$u = -\tfrac{1}{2}e^x + C_1 e^{-x}$$

But $u = \frac{1}{y}$ $\qquad \dfrac{1}{y} = \dfrac{-e^{2x} + C}{2e^x}$

$$\boxed{y = \frac{2e^x}{C - e^{2x}}}$$

OR $y = \left(Ce^{-x} - \tfrac{1}{2}e^x\right)^{-1}$

6. $(y^2 + yx)\,\boxed{dx} + x^2\,dy = 0$

This is a homogeneous equation

Suppose we replace $x \to tx$ and $y \to ty$.

Then we have $(t^2y^2 + ty\,tx)dx + t^2x^2dy = 0$

$t^2[(y^2+yx)dx + x^2dy] = 0$

t^α

$u = \frac{y}{x}$ $\quad v = \frac{x}{y}$

Substitution is either $y = ux$ or $x = vy$

If $y = ux$, then $\quad$ If $x = vy$, then

$dy = u\,dx + x\,du$ $\quad$ $dx = v\,dy + y\,dv$

6. $(y^2 + yx)\,dx + x^2\,dy = 0$

$(u^2x^2 + ux^2)dx + x^2(u\,dx + x\,du) = 0$

$u^2dx + u\,dx + u\,dx + x\,du = 0$

$(u^2 + 2u)dx + x\,du = 0$

$$\int \frac{1}{x}\,dx + \int \frac{1}{u^2+2u}\,du = \int 0\,dx$$

$$\frac{1}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2} \Rightarrow 1 = A(u+2) + Bu$$

$u=0 \quad A = \frac{1}{2}$

$u=-2 \quad B = -\frac{1}{2}$

$$\int \frac{1}{x}\,dx + \int \left(\frac{1/2}{u} - \frac{1/2}{u+2}\right) du = \int 0\,dx$$

$$\ln|x| + \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2|$$

$$\frac{1}{2}\left(\ln|u| - \ln|u+2|\right)$$

$$e^{\ln\left|x\sqrt{\frac{u}{u+2}}\right|} = e^{\ln|C_1|}$$

$$x\sqrt{\frac{u}{u+2}} = C_1$$

$C = C_1^{\,2}$

$$\frac{u}{u+2} = \frac{C}{x^2}$$

But $u = y/x$, so $\dfrac{y/x}{y/x+2}\dfrac{x}{x} = \dfrac{C}{x^2}$

$$\frac{y}{y+2x} = \frac{C}{x^2} \Rightarrow \boxed{x^2 y = C(y+2x)}$$

$$x^2\,dy + \sqrt{x^3y + xy^3}\,dx = 0$$

$$x \Rightarrow tx \qquad y \Rightarrow ty$$

$$t^2x^2\,dy + \sqrt{t^4x^3y + t^4xy^3}\,dx = 0$$

$$t^2\,(x^2)\,dy + t^2\sqrt{x^3y + xy^3}\,dx = 0$$

Given $M(x,y)\,dx + N(x,y)\,dy = 0$ $(*)$

If $M(tx,ty)\,dx + N(tx,ty)\,dy = 0$

becomes $t^{\alpha}\left(M(x,y)\,dx + N(x,y)\,dy\right) = 0$,

then $(*)$ is homogeneous of order α.

$$f(x,y)\,\frac{dy}{dx} = 0$$

$$x^{3/2}\,dx + \sqrt{x}\,y\,dy = 0$$

$$\underbrace{(tx)^{3/2}}_{t^{3/2}}\,dx + \underbrace{\sqrt{tx}\;ty\,dy}_{t^{3/2}} = 0$$

20. $3(1+t^2)\dfrac{dy}{dt} = 2ty(y^3-1)$

$$\underbrace{3(1+t^2)\frac{dy}{dt}}_{\text{linear}} = 2ty^4 - \underbrace{2ty}_{\text{linear}}$$

$$3(1+t^2)\frac{dy}{dt} + 2ty = 2ty^4 \qquad \checkmark \text{ Bernoulli } n=4$$

sub: $u = y^{1-4} = y^{-3}$ $\qquad (y^{-3})^{-1/3} = u^{-1/3}$

$$y = u^{-1/3}$$

$$\frac{dy}{dt} = -\frac{1}{3}u^{-4/3}\frac{du}{dt}$$

$$\frac{dy}{dt} + \frac{2t}{3(1+t^2)}y = \frac{2ty^4}{3(1+t^2)}$$

$$-\frac{1}{3}u^{-4/3}\frac{du}{dt} + \frac{2t}{3(1+t^2)}u^{-1/3} = \frac{2tu^{-4/3}}{3(1+t^2)}$$

mult by $-3u^{4/3}$

$$\frac{du}{dt} - \frac{2t}{1+t^2}u = -\frac{2t}{1+t^2} \qquad \text{(linear)}$$

$$\mu = e^{-\int \frac{2t}{1+t^2}dt} = e^{-\ln(1+t^2)} = \frac{1}{1+t^2}$$

$$\frac{d}{dt}\left(\frac{1}{1+t^2}u\right) = -\frac{2t}{(1+t^2)^2}$$

$$\frac{1}{1+t^2}u = \frac{1}{1+t^2} + C$$

$$u = 1 + C(1+t^2)$$

$$\boxed{y^{-3} = 1 + C(1+t^2)}$$

Summary

Homogeneous eqn

Sub: $y = ux$ OR $x = vy$

$dy = x\,du + u\,dx$ $\quad dx = v\,dy + y\,dv$

all terms have the same "degree"

Bernoulli:

$$\underbrace{\frac{dy}{dx} + P(x)\,y = f(x)}_{\text{linear}}\,y^n \qquad u = y^{1-n}$$

linear substitution:

$$\frac{dy}{dx} = f(ax+by+c) \qquad u = ax+by+c$$

2.4: Exact if $M\,dx + N\,dy = 0$ is such that $M_y = N_x$